Family Name:__________________**Given Name:**__________________ **I.D.#**_________

MAT3320 Assignment 4

Total: 10 marks. Due date: Tuesday, July 18, on or before 4:00pm.

In MATH Department (585 King Edward), there is a Drop-Box. You need to put your assignment into the box **on or before 4:00pm** on the due date. Late assignments will not be accepted.

1. (2 marks) The solution of Laplace's equation $u_{xx} + u_{yy} = 0$, $0 < x < L, 0 < y < M$ with the boundary conditions $u(x, M) = f(x), u(x, 0) = u(0, y) = u(L, y) = 0$ is given by

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

Find the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within

$$R = \{(x, y) : 0 < x < 3, 0 < y < 2\}$$

with

$$BC: \quad u(x, 0) = 0, u(x, 2) = x - 3, u(0, y) = 0, u(3, y) = 0.$$

Solution: $L = 3,\ M = 2.$ $u(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi y}{3}\right) \sin\left(\frac{n\pi x}{3}\right)$.

$$u(x, 2) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{2n\pi}{3}\right) \sin\left(\frac{n\pi x}{3}\right) = x - 3.$$

Thus

$$b_n \sinh\left(\frac{2n\pi}{3}\right) = \frac{2}{3}\int_0^3 (x-3)\sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\left[(x-3)\left(-\frac{3}{n\pi}\cos\frac{n\pi x}{3}\right) + \left(\frac{3}{n\pi}\right)^2 \sin\frac{n\pi x}{3}\right]\Bigg|_0^3 = -\frac{6}{n\pi}.$$

$$b_n = \frac{-6}{n\pi \sinh\left(\frac{2n\pi}{3}\right)}.$$

Hence

$$u(x,y) = \sum_{n=1}^{\infty} \frac{-6}{n\pi \sinh\left(\frac{2n\pi}{3}\right)} \sinh\left(\frac{n\pi y}{3}\right) \sin\left(\frac{n\pi x}{3}\right).$$

2. (3 marks) The solution of $u_{tt} = c^2(u_{xx} + u_{yy}), \quad (x, y) \in R = [0, a] \times [0, b], t > 0$, subject to $BC : u(x, y, t) = 0$ for $t > 0$ and $(x, y) \in \partial R$ (boundary of R), is

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (A_{mn} \cos[\lambda_{mn} t] + B_{mn} \sin[\lambda_{mn} t]) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

where $\lambda_{mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

Solve $u_{tt} = 25(u_{xx} + u_{yy}), \quad (x, y) \in R = [0, 3] \times [0, 2], t > 0$, subject to

$BC : u(x, y, t) = 0$ for $t > 0$ and $(x, y) \in \partial R$,

$ICs : u(x, y, 0) = 0, u_t(x, y, 0) = \pi \sin(3\pi x) \sin(4\pi y), (x, y) \in R.$

Solution: $c = 5$, $a = 3$, $b = 2$, $\lambda_{mn} = 5\pi \sqrt{\frac{m^2}{9} + \frac{n^2}{4}}$.

$$u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (A_{mn}) \sin \frac{m\pi x}{3} \sin \frac{n\pi y}{2} = 0 \Rightarrow$$

$$A_{mn} = 0, \forall m, n.$$

$$u_t(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (B_{mn} \lambda_{mn}) \sin \frac{m\pi x}{3} \sin \frac{n\pi y}{2} = \pi \sin(3\pi x) \sin(4\pi y) \Rightarrow$$

$$\lambda_{9,8} B_{9,8} = \pi, \quad B_{mn} = 0, (m, n) \neq (9, 8).$$

$$\lambda_{9,8} = 5\pi \sqrt{\frac{9^2}{9} + \frac{8^2}{4}} = 25\pi, \Rightarrow B_{9,8} = \frac{1}{25}.$$

$$u(x, y, t) = \frac{1}{25} \sin(25\pi t) \sin(3\pi x) \sin(4\pi y).$$

3. (2 marks) Find the solution $u(r, \varphi)$ of

$$u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} + \frac{\cot \varphi}{r^2} u_\varphi = 0, \quad r < 4,$$

such that $u(0, \varphi)$ is bounded, and $u(4, \varphi) = \cos(2\varphi)$.

Solution:

$$u(r, \varphi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \varphi),$$

$$u(4, \varphi) = \sum_{n=0}^{\infty} A_n 4^n P_n(\cos \varphi) = \cos(2\varphi) = 2\cos^2(\varphi) - 1$$

$$= \frac{4}{3} P_2(\cos \varphi) - \frac{1}{3} P_0(\cos \varphi)$$

$$\Rightarrow A_2 4^2 = \frac{4}{3}, A_0 4^0 = -\frac{1}{3}, A_n 4^n = 0, n \neq 0, 2.$$

$$\Rightarrow A_2 = \frac{1}{12}, A_0 = -\frac{1}{3}, A_n = 0, n \neq 0, 2.$$

Thus

$$u(r, \varphi) = -\frac{1}{3} + \frac{1}{12} r^2 P_2(\cos\varphi).$$

4. (3 marks) Evaluate

$$\int_0^3 x^5 J_2(2x)dx.$$

Remark. Your solution should be a linear combination of $J_3(6)$ and $J_4(6)$.

Solution: Let $t = 2x$, then

$$\int_0^3 x^5 J_2(2x)dx = 2^{-6}\int_0^6 t^5 J_2(t)dt = 2^{-6}\int_0^6 t^2[t^3 J_2(t)]dt = 2^{-6}\int_0^6 t^2 d[t^3 J_3(t)]$$

$$= 2^{-6} t^2 [t^3 J_3(t)]|_0^6 - 2^{-6}\int_0^6 2t[t^3 J_3(t)]dt = 2^{-1}3^5 J_3(6) - 2^{-5}\int_0^6 t^4 J_3(t)dt$$

$$= 2^{-1}3^5 J_3(6) - 2^{-5} t^4 J_4(t)|_0^6 = 2^{-1}3^5 J_3(6) - 2^{-5}6^4 J_4(6)$$

$$= 2^{-1}3^5 J_3(6) - 2^{-1}3^4 J_4(6) = \frac{243}{2} J_3(6) - \frac{81}{2} J_4(6).$$